Lecture 12 (10/20/21)

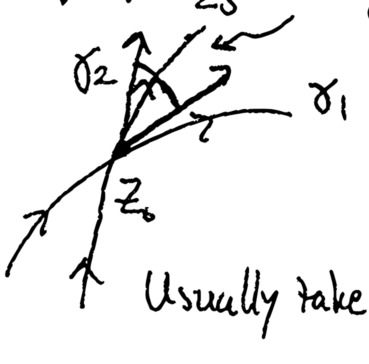
- Cauchy-Riemann Equations and harmonic functions from Lecture 11 notes.

Conformal maps.

Def. (1) A path or curve in $G \subseteq \mathbb{C}$ is a continuous map $\gamma : [a,b] \to G$. The curve is $\mathcal{C}^1$ if γ is $\mathcal{C}^1$ as map and $\gamma'(t) \neq 0$, piecewise (pw) $\mathcal{C}^1$ if $[a,b] = \bigcup_{k=1}^{n} [a_{k-1}, a_k]$, $a_0 = a$ and $a_n = b$, s.t. $\gamma : [a_{k-1}, a_k]$ is $\mathcal{C}^1$.

If γ_1, γ_2 are $\mathcal{C}^1$ paths intersecting at z_0 (WLOG $\exists\, t_0$ s.t. $\gamma_1(t_0) = \gamma_2(t_0) = z_0$)

then $\measuredangle(\gamma_1, \gamma_2)_{z_0} = \arg \gamma_2'(t_0) - \arg \gamma_1'(t_0)$



Ambiguity of "arg" and fact that there are 2 angles should not cause confusion.

Usually take "arg" s.t. $-\pi < \measuredangle \leq \pi$

If f is analytic in G, $\gamma : [a,b] \to G$ a (p.w.) $\mathcal{C}^1$ path, then $\sigma = f \circ \gamma$ is a path in $f(G)$. If $f' \neq 0$, then σ is (p.w.) $\mathcal{C}^1$. Note: We are considering f as map $G \subseteq \mathbb{C} \to f(G) \subseteq \mathbb{C}$.

Thm 1. Let γ_1, γ_2 be paths in G, intersecting at $z_0 \in G$. If f is analytic in G, $f'(z_0) \neq 0$, then $\measuredangle(\gamma_1, \gamma_2)_{z_0} = \measuredangle(\sigma_1, \sigma_2)_{f(z_0)}$, where $\sigma_j = f \circ \gamma_j$.

Pf. Chain rule $\Rightarrow \sigma_j'(t_0) = f'(z_0)\gamma_j'(t_0)$. Multiplication by cplx number $f'(z_0) \neq 0$ is scaling length by $|f'(z_0)|$ and rotation by $\arg f'(z_0)$. Since $\measuredangle$ are preserved under rotation (if understood properly), the conclusion follows. $\boxtimes$

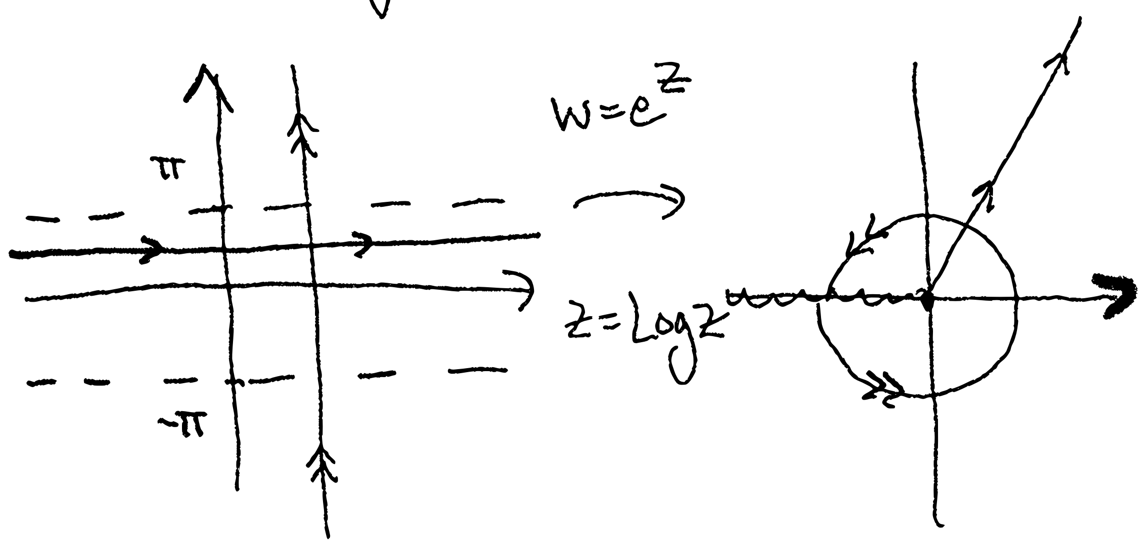
Rem. A $\mathcal{C}^1$-map f is conformal if it preserves $\measuredangle$ (angle + "orientation").

sign of angle between γ_1, γ_2.

By CR eq's $f : G \to \mathbb{C}$ is conformal in G $\iff$ f is analytic and $f' \neq 0$. Recall f is anal. $\iff$ $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

and 2×2 matrix A preserves $\measuredangle$ $\iff$ $A = rU$, $r > 0$ and $U \in SO(\mathbb{R}^2)$.

<u>Ex.</u> e^z, $\operatorname{Log} z$ are conformal maps.



Möbius tranformations.

Def (2). A <u>Möbius transformation</u> is a map $T(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$.

- T is defined in $\mathbb{C}\setminus\{-d/c\}$ and is conformal there (anal. + $T' \neq 0$). We usually consider $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$, where $T(-d/c) = \infty$ and $T(\infty) = \lim_{z\to\infty} T(z)$ (which exists as an element of $\mathbb{C}_\infty$). In this way, T is a <u>continuous</u> map $\mathbb{C}_\infty \to \mathbb{C}_\infty$, conformal on $\mathbb{C}\setminus\{-d/c\}$.

- T is a bijection $\mathbb{C}_\infty \to \mathbb{C}_\infty$ and T^{-1} is also Möbius:

$$w = \frac{az+b}{cz+d} \iff z = \frac{dw-b}{-cw+a}.$$

(consistent w/ def's of $T(\infty) = \frac{a}{c}$, $T(-\frac{d}{c}) = \infty$).

- $T' = \frac{ad-bc}{(cz+d)^2}$.